Worksheet ∞

Name: ______________________ Score: ________

1 Topics

1. For each topic, explain: what is it? how do you find it or do it? what does it mean? What is its use and what does it do? Give examples.

   - row echelon form and reduced row echelon form ((R)REF)
   - How to solve a system of linear equations. When is a linear system of equations consistent or inconsistent?
   - linear combinations
   - the span $\text{Span}(v_1, \ldots, v_p)$ of a set of vectors (and how to find a basis for it)
   - when is a vector $\vec{b}$ in the span $\text{Span}\,(v_1, \ldots, v_p)$ of some other vectors?
   - When does a set of vectors $\{v_1, \ldots, v_p\}$ span all of $\mathbb{R}^n$? When is it linearly independent? How do you check and what does it mean?
   - linear functions
   - one to one/onto/isomorphism. How to check if a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is one of them?
   - The matrix of a linear map $f : \mathbb{R}^n \to \mathbb{R}^m$. Give examples, such as rotation, projection, the identity matrix, etc. Draw pictures.
   - matrix multiplication, in particular multiplying a vector by a matrix and composition of linear functions
   - subspaces
   - bases
   - the null space, column space of a matrix A. (and how to find bases for them)
   - inverses (general formula and the 2x2 case). How to use A^{-1} to solve $A\vec{x} = \vec{b}$. When is a matrix invertible?
   - determinants (general formula and the 2x2 case)
   - Change of basis on $\mathbb{R}^n$.
   - General vector spaces V. Give examples: $\mathbb{P}_2, \mathbb{P}_n, \mathbb{P}$, continuous/differentiable functions $f : \mathbb{R} \to \mathbb{R}$, the vector space of all 3x2 matrices, etc.
   - the coordinates of a vector $\vec{v}$ in V in a basis B for V.
   - The matrix of a linear map $f : V \to W$ between general vector spaces. (need to choose bases for V and W) Give examples, such as the derivative, integral, or indefinite integral of the vector space of polynomials
   - eigenvalues and eigenvectors

- diagonalizing matrices
- Matrix differential equations $\vec{x}'(t) = A\vec{x}(t)$ and initial value problems where $\vec{x}(0)$ is given
- Complex numbers $\mathbb{C}$ and complex eigenvalues/vectors
- The dot product
- The length of a vector $\vec{u}$ and the angle between two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.
- orthogonal and orthonormal sets $\{v_1, \ldots, v_p\}$ of vectors
- The orthogonal complement $W^\perp$ of a subspace W of $\mathbb{R}^n$. How to find a basis for it? $(\operatorname{Col} A)^\perp = \operatorname{Nul}(A^T)$. (This and least squares were our only uses of the transpose)
- the Gram Schmidt process
- Least squares solutions to $A\vec{x} = \vec{b}$. (multiply by A^T: $A^T A\vec{x} = A^T\vec{b}$.)
- Anything I forgot?

Some problems:

2. Find a basis of solutions for the Matrix differential equation

$$\begin{aligned} x_1' &= 2x_1 - 2x_2 \\ x_2' &= 4x_1 - 4x_2 \end{aligned} \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}$$

Eigenvalues and vectors:

$$0 : \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad -2 : \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3. Find the angle between the two vectors. Find the length of each vector.

- $$\begin{bmatrix} 13 \\ -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ 13 \end{bmatrix}$$
- $$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} \sqrt{6} + \sqrt{2} \\ \sqrt{6} - \sqrt{2} \end{bmatrix}$$
- $$\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}, \qquad \begin{bmatrix} 0 \\ -13 \\ 0 \end{bmatrix}$$

4. Is the vector $\vec{b}$ in the span of the other vectors?

- $$\vec{b} = \begin{bmatrix} -22 \\ 5 \\ -5 \end{bmatrix}$$

  $$\begin{bmatrix} -6 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 10 \\ 1 \\ -1 \end{bmatrix}$$

  The vector is in the span of the other vectors.

  $$\begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- $$\vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

  $$\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ -5 \end{bmatrix} \quad \begin{bmatrix} -10 \\ 25 \\ -5 \end{bmatrix}$$

  The vector is not in the span of the other vectors.

  $$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. Find the determinants

- $$\begin{bmatrix} 45 & 12 & 12 \\ -48 & -13 & -12 \\ -12 & -3 & -4 \end{bmatrix}$$

  $$\det A = 0$$

- $$\begin{bmatrix} 6 & 38 & 46 \\ 0 & 2 & 4 \\ -1 & -8 & -11 \end{bmatrix}$$

  Determinant:

  $$\det A = 0$$

6. Find the inverses

   - $$\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$$
     $$A^{-1} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$$
   - $$\begin{bmatrix} -4 & -3 \\ -1 & -1 \end{bmatrix}$$
     Inverse:
     $$A^{-1} = \begin{bmatrix} -1 & 3 \\ 1 & -4 \end{bmatrix}$$

7. Are the vectors
   $$\begin{bmatrix} 1 \\ -1 \\ -3 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -7 \\ 3 \\ -6 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ -6 \\ 4 \\ -2 \end{bmatrix},$$
   linearly independent?

   Yes

8. Do the vectors
   $$\begin{bmatrix} -2 \\ -2 \\ -9 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 4 \\ 18 \end{bmatrix} \quad \begin{bmatrix} -4 \\ -8 \\ -20 \end{bmatrix}$$
   span $\mathbb{R}^3$?

   The set does not span all of $\mathbb{R}^3$.

9. Find a basis for the null space and column space of the matrix
   $$\begin{bmatrix} 0 & -9 & 9 & 36 \\ -2 & 2 & -10 & -6 \\ 5 & 4 & 16 & -21 \end{bmatrix}$$
   Reduced Row Echelon Form:
   $$\begin{bmatrix} 1 & 0 & 4 & -1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
   Basis:
   $$\begin{bmatrix} -4 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix},$$

10. Find a basis for the span of the vectors:

$$\begin{bmatrix}4\\0\\-2\end{bmatrix} \quad \begin{bmatrix}0\\-2\\-5\end{bmatrix} \quad \begin{bmatrix}8\\2\\1\end{bmatrix} \quad \begin{bmatrix}-4\\3\\2\end{bmatrix}$$

$$\begin{bmatrix}4\\0\\-2\end{bmatrix} \quad \begin{bmatrix}0\\-2\\-5\end{bmatrix} \quad \begin{bmatrix}-4\\3\\2\end{bmatrix}$$

11. Find the change of basis matrix to get from B to C:

$$B = \begin{bmatrix}-5\\1\end{bmatrix}, \quad \begin{bmatrix}17\\1\end{bmatrix},$$

$$C = \begin{bmatrix}-5\\1\end{bmatrix}, \quad \begin{bmatrix}-2\\-4\end{bmatrix}, \quad .$$

Solution:

$$\underset{C\leftarrow B}{P} = \begin{bmatrix}1 & -3\\0 & -1\end{bmatrix}$$

12. Let W be the subspace spanned by the vectors

$$\begin{bmatrix}2\\0\\3\\-9\end{bmatrix}, \quad \begin{bmatrix}-1\\0\\-3\\12\end{bmatrix}.$$

Find a basis for the orthogonal complement $W^\perp$ of W. Solution:

Let A be the matrix with columns the above vectors:

$$\begin{bmatrix}2 & -1\\0 & 0\\3 & -3\\-9 & 12\end{bmatrix}$$

Reduced row echelon form (RREF) of A^T:

$$\begin{bmatrix}1 & 0 & 0 & 3\\0 & 0 & 1 & -5\end{bmatrix}$$

Basis for $W^\perp = (\text{Col}A)^\perp = \text{Nul}A^T$:

$$\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}-3\\0\\5\\1\end{bmatrix}.$$

13. Find the eigenvalues and eigenvectors:

$$\begin{bmatrix} 7 & -12 \\ 2 & -3 \end{bmatrix}$$

$$1 : \begin{bmatrix} 2 \\ 1 \end{bmatrix} 3 : \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

14. Diagonalize the matrix:

$$\begin{bmatrix} -5 & 9 \\ -6 & 10 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 9 \\ -6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$$